PMATH 336: INTRODUCTION TO GROUP THEORY WITH APPLICATIONS
NOTES FOR WEEK 11

INSTRUCTOR: ARUNDHATHI KRISHNAN

12. Group Actions and Burnside's Lemma

12.1. Group actions.

Definition 12.1.1. A (left) group action of a group G on a set X is a function $\varphi : G \times X \to X$ satisfying the following properties:

(i) (Compatibility) $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ for all $g, h \in G$ and for all $x \in X$.
(ii) (Identity) $\varphi(e, x) = x$ for all $x \in X$.

In this case, the group G is said to act on the set X (via the group action φ).

In fact, the above definition simply states that for each $g \in G$, there exists a map $\varphi(g, \cdot)$ from X to X, which we will show shortly is actually a bijection or permutation of the set X. Further, we will show that the map $g \mapsto \varphi(g, \cdot)$ is a group homomorphism from G to a permutation group on the set X, and conversely, that any group homomorphism from G to a permutation group on the set X is obtained via a group action.

We have already encountered many examples of group actions. We will look at them after seeing some other simple examples.

Example 12.1.2.

(i) Define $\varphi_i : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ for $i = 1, 2$, by $\varphi_1(a, (x, y)) = (x + a, y)$ and $\varphi_2(b, (x, y)) = (x, y + b)$. Then φ_1 and φ_2 are group actions. They are the actions of horizontal and vertical translations respectively on $\mathbb{R}^2$.

We check that φ_1 is a group action:

(a) $\varphi_1(a_1 + a_2, (x, y)) = (x + a_1 + a_2, y) = \varphi_1(a_1, (x + a_2, y)) = \varphi(a_1, \varphi_1(a_2, (x, y))$ for all $a_1, a_2 \in \mathbb{R}$ and for all $(x, y) \in \mathbb{R}^2$.
(b) $\varphi(0, (x, y)) = (x + 0, y) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$.

Note that here G is the Abelian group $\mathbb{R}$ with the operation of addition, and $X = \mathbb{R}^2$.

(ii) Let $G = \{e, a\}$ and $X = \mathbb{C}$. Then G acts on X by $\varphi : G \times X \to X$ given by $\varphi(e, x + iy) = x + iy$ and $\varphi(a, x + iy) = x - iy$. (Verify!)

(iii) Every subgroup H of a group G (including G itself) acts on G by left multiplication. That is, $\varphi(h, x) = hx$ for all $h \in H$ and for all $x \in G$ is a group action.

To see this, observe that $\varphi(h_1h_2, x) = (h_1h_2)x = h_1(h_2x) = \varphi(h_1, h_2x) = \varphi(h_1, \varphi(h_2, x))$ and $\varphi(e, x) = x$ for all $h_1, h_2 \in H$ and $x \in G$.

If $H = G$, we get for each $g \in G$ and $x \in G$, $\varphi(g, x) = gx = L_g(x)$, where L_g is the function of left multiplication on G. Recall from the proof of Cayley's theorem (Theorem 6.2.1) that L_g is a bijection or permutation of G for each $g \in G$ and the map $g \mapsto L_g$ is a group homomorphism.

(iv) Let G be a group and H a subgroup of G. Let $\mathcal{L}$ be the set of cosets of H in G, and the map $L_g : \mathcal{L} \to \mathcal{L}$ given by $L_g(xH) = gxH$. Show that $\varphi : G \times \mathcal{L} \to \mathcal{L}$ given by $\varphi(g, xH) = L_g(xH) = gxH$ is a group action.

(v) A subgroup H of a group G acts on G by conjugation $\varphi(h, x) = hxh^{-1}$.

(vi) Let $X = \{1, \ldots, n\}$ and $G = S_n$. Then G acts on X by $\varphi(\alpha, i) = \alpha(i)$. In fact, a group action is simply a generalisation of the case that the group itself is a permutation group like in this example.

Theorem 12.1.3. *Let G be a group acting on the set X.*

(i) For every $g \in G$, the mapping $\varphi_g : X \to X$ defined by $\varphi_g(x) = \varphi(g, x)$ for all $x \in X$, is a permutation of X.

(ii) The mapping $g \mapsto \varphi_g$ is a group homomorphism between G and a group of permutations of X.

Proof.

(i) We will show that $\varphi_{g^{-1}}$ is the inverse of each φ_g, so that the latter (and the former!) is a bijection. $\varphi_{g^{-1}}\varphi_g(x) = \varphi(g^{-1}, \varphi(g, x)) = \varphi(g^{-1}g, x) = \varphi(e, x) = x$ for each $x \in X$. Similarly, $\varphi_g\varphi_{g^{-1}}(x) = x$ for all $x \in X$.

(ii) Let $g, h \in G$ and $x \in X$. Then $\varphi_{gh}(x) = \varphi(gh, x) = \varphi(g, \varphi(h, x)) = \varphi(g, \varphi_h(x)) = \varphi_g\varphi_h(x)$, so that $\varphi_{gh} = \varphi_g\varphi_h$. This shows that $g \mapsto \varphi_g$ is a homomorphism.

□

The converse of the above theorem is also true.

Theorem 12.1.4. *Let G be a group, X be a set and $\mathcal{S}$ be a permutation group of X. If $\psi : G \to \mathcal{S}$ is a group homomorphism, then $\varphi : G \times X \to X$ given by $\varphi(g, x) = \psi(g)(x)$, for all $g \in G$ and $x \in X$, is a group action of G on X.*

The theorem gives in particular that $\psi(g) = \varphi_g$ for every $g \in G$.

Proof. We check that the two conditions of a group action are satisfied:

(i) $\varphi(e, x) = \psi(e)x = x$ as ψ by virtue of being a homomorphism must take the identity of G to the identity permutation.

(ii) $\varphi(gh, x) = \psi(gh)(x) = \psi(g)\psi(h)(x) = \psi(g)(\varphi(h, x)) = \varphi(g, \varphi(h, x))$ as ψ is a homomorphism.

□

With the above two theorems, we have a one-to-one correspondence between homomorphisms from a group G to a permutation group of a set X, and group actions of G on X.

12.2. **Burnside's lemma.** We will now use the machinery of group actions to prove an important result in counting applications. While commonly attributed to Burnside, it was actually proved by Frobenius, and is sometimes referred to as the Cauchy-Frobenius lemma, with Pólya also often being named in connection to it.

In order to arrive at this result, let us first recall the Orbit-Stabilizer theorem (Theorem 7.3.5). We will slightly restate the definitions of the stabilizer and orbit in the context of group actions.

Definition 12.2.1. Let $\varphi : G \times X \to X$ be a group action. The stabilizer of an element $x \in X$ in G is defined as the following set:

$$\mathrm{stab}_G^{\varphi}(x) = \{g \in G \mid \varphi(g, x) = \varphi_g(x) = x\}.$$

Note that the above is only a small restatement of Definition 7.3.1. In that case, G itself was a group of permutations, whereas here there is a homomorphism from G to a group of permutations given by $g \mapsto \varphi_g$.

Definition 12.2.2. Let $\varphi : G \times X \to X$ be a group action. The orbit of an element $x \in X$ under G is defined as the following set:

$$\mathrm{orb}_G^{\varphi}(x) = \{\varphi_g(x) \mid g \in G\}.$$

The Orbit Stabilizer theorem in this case is translated to the following.

Theorem 12.2.3 (Orbit Stabilizer theorem for Group Actions)**.** *Let G be a finite group, X a set, and $\varphi : G \times X \to X$ be a group action. Then for any $x \in X$, $|\,G\,| = \left|\,\mathrm{orb}_G^{\varphi}(x)\,\right| \left|\,\mathrm{stab}_G^{\varphi}(x)\,\right|$.*

We define another subset of X, namely the fixed points of X under a permutation φ_g.

Definition 12.2.4. Let $\varphi : G \times X \to X$ be a group action. For $g \in G$, let X^g denote the set of elements of X that are fixed by φ_g:

$$X^g = \{x \in X \mid \varphi_g(x) = \varphi(g, x) = x\}.$$

Remark 12.2.5.

(i) Let $\varphi : G \times X \to X$ be a group action. For an element $x \in X$, $\left|\,\mathrm{orb}_G^{\varphi}(x)\,\right| = 1$ if and only if $\mathrm{orb}_G^{\varphi}(x) = \{x\}$ if and only if $x \in X^g$ for all $g \in G$.

(ii) If a and b are in the same orbit, then the orbits of a and b are equal. This gives immediately that the relation $a \sim b$ if $a \in \mathrm{orb}_G^{\varphi}(b)$ is an equivalence relation.

(iii) Further, by the orbit stabilizer theorem, if $a \sim b$, then the cardinalities of $\mathrm{stab}_G^{\varphi}(a)$ and $\mathrm{stab}_G^{\varphi}(b)$ are the same.

We are now ready to state and prove the main result of this subsection, which is essentially a theorem that gives us a way to count the number of orbits of a given group action.

Theorem 12.2.6 (Burnside's lemma/ Orbit Counting theorem/ Cauchy-Frobenius lemma)**.** *Let $\varphi : G \times X \to X$ be a group action, where G is a finite group and X is a set. Then the number of distinct orbits of elements of X is given by*

$$\frac{1}{|\,G\,|} \sum_{g \in G} |\,X^g\,|.$$

Proof. Let n be equal to the number of pairs (g, x) where $\varphi_g(x) = \varphi(g, x) = x$. This can be counted in two ways. One is by fixing $g \in G$ first, and the other, by fixing $x \in X$ first.

For each $g \in G$, the number of pairs such that $\varphi_g(x) = x$ is equal to $|\,X^g\,|$, so that $n = \sum_{g \in G} |\,X^g\,|$. On the other hand, for each $x \in X$, the number of such pairs is equal to $\left|\,\mathrm{stab}_G^{\varphi}(x)\,\right|$, so that $n = \sum_{x \in X} \left|\,\mathrm{stab}_G^{\varphi}(x)\,\right|$.

For each $x \in X$, summing $\left|\,\mathrm{stab}_G^{\varphi}(t)\,\right|$ over $t \in \mathrm{orb}_G^{\varphi}(x)$ gives

$$\begin{aligned}\sum_{t \in \mathrm{orb}_G^{\varphi}(x)} \left|\,\mathrm{stab}_G^{\varphi}(t)\,\right| &= \left|\,\mathrm{orb}_G^{\varphi}(s)\,\right| \left|\,\mathrm{stab}_G^{\varphi}(x)\,\right| \quad \text{(as for each } t \in \mathrm{orb}_G^{\varphi(x)}, \left|\,\mathrm{stab}_G^{\varphi}(t)\,\right| \text{ is the same)} \\ &= |\,G\,| \quad \text{(by the Orbit Stabilizer theorem).}\end{aligned}$$

That is, the sum of $\left|\operatorname{stab}_g^{\varphi}(t)\right|$ where t varies over a fixed orbit is $|\,G\,|$. Hence,

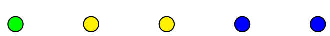
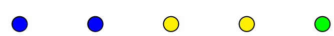
$$\begin{aligned}
\sum_{g\in G} |\,X^g\,| = n = \sum_{x\in X} \left|\operatorname{stab}_G^{\varphi}(x)\right| \\
&= \sum_{\text{distinct orbits in } X} \;\; \sum_{t\in \operatorname{orb}_G^{\varphi}(x)} \left|\operatorname{stab}_G^{\varphi}(t)\right| \\
&= \sum_{\text{distinct orbits in } X} |\,G\,| \\
&= |\,G\,| \times \text{number of orbits.}
\end{aligned}$$

□

12.3. **Counting Applications.** We now see how to apply Burnside's lemma (or the Lemma that is not Burnside's if you will!) to various counting problems.

Example 12.3.1. Suppose we have a string of n beads where each bead can have t colours. There are t^n such configurations. As the string can be flipped over, we have certain repetitions. This can be explained using the tool of a group acting on a set. Let X be the set of all possible configurations. As the only symmetry possible is about the centre of the string (achieved by flipping the string over), the group we consider is $G = \mathbb{Z}_2$. Here 0 will act on X by doing nothing, and 1 acts by flipping the string.

Say, for example that $n = 5$ and $t = 3$, with colours say, green, yellow and blue. Some examples of configurations which are the same as each other (on flipping over) are

 and 

In the language of orbits, the two configurations above are equivalent via the relation of belonging to the same orbit.

We want to count the number of *distinct* configurations, or in other words, the number of distinct orbits. By Theorem 12.2.6,

$$\text{Number of orbits} = \frac{1}{|\,\mathbb{Z}_2\,|} \sum_{g\in\mathbb{Z}_2} X^g.$$

For $g = 0$, $X^g = X$ as every configuration is fixed by doing nothing. On the other hand, the number of fixed points of 1 (flipping the string over) are determined by one half of the string (as the other half must be the same by symmetry). This depends on whether n is even or odd. If n is even, then we have $t^{\frac{n}{2}}$ fixed points, and if n is odd, we have $t^{\frac{n+1}{2}}$ fixed points. So we get

$$\text{Number of orbits} = \frac{1}{2}(t^n + t^{\frac{n}{2}}),$$

if n is even and

$$\text{Number of orbits} = \frac{1}{2}(t^n + t^{\frac{n+1}{2}}),$$

if n is odd.

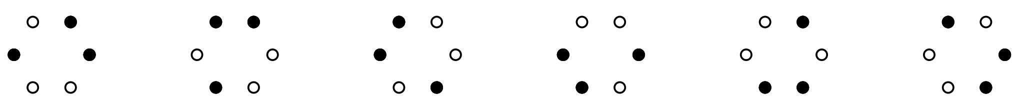
In both cases, if $t = 1$, then for any n we get only one orbit. This is as expected since only one distinct string of n beads can be made if the beads are all of the same colour.

Example 12.3.2. Suppose we want to count the number of ways in which the six vertices of a hexagon can be coloured so that three are black and three are white. There are $\binom{6}{3} = 20$ ways to do this. However, if the hexagons were actually ceramic tiles, say, there would clearly be some repetitions as some of the patterns can be obtained from the remaining ones by rotation.

The 20 possibilities are given below, where the figures on each line can be obtained from the others on the same line by rotation.

(a)


(b)


(c)


(d)


We will now take X to be the set of all 20 possibilities given above and G to be the group of rotational symmetries of the hexagon $\{r_0, r_1, \dots, r_5\}$ (with notation borrowed from the dihedral group D_6). Then G acts on X by rotating the diagrams, and the lines a, b, c and d of diagrams that can be obtained from each other by rotation describe precisely the distinct orbits of the group action. In other words, a diagram that can be obtained from another by a rotation is equivalent to it via the equivalence relation of belonging to the same orbit.

We can now use Burnside's lemma to verify that the number of orbits of this group action is indeed 4.

$$\text{Number of orbits} = \frac{1}{|\,G\,|} \sum_{g \in G} |\,X^g\,|.$$

Here $|\,G\,| = 6$. We calculate X^g for each $g \in G$ below.

Element	$\lvert X^g \rvert$
r_0	20
r_1	0
r_2	2
r_3	0
r_4	2
r_5	0

Here, we see that r_2 and r_4 fix exactly the two elements on line (d), r_0 fixes all 20 elements and the remaining rotations do not fix any of the figures.

Hence

$$\text{Number of orbits} = \frac{1}{6}(20 + 2 + 2) = 4.$$

What happens if we consider these patterns not on a hexagonal ceramic tile, but instead on a necklace? In this case, all the figures on line (b) would be equivalent to those on line (c) as the necklace can also be turned over. So the number of distinct configurations would only be 3. To understand this in terms of orbits, we see that G in this case is all of D_6, as the necklaces remain unchanged on reflections as well. In this case we get

Element	$\lvert X^g \rvert$
r_0	20
r_1	0
r_2	2
r_3	0
r_4	2
r_5	0
s_0	4
s_1	0
s_2	4
s_3	0
s_4	4
s_5	0

and

$$\text{Number of orbits} = \frac{1}{12}(20 + 2 + 2 + 4 + 4 + 4) = 3.$$

Example 12.3.3. Now suppose that the necklace consists of 6 beads, where each bead can be one of t colours. How many distinct figures are possible?

Here again, as the necklace can be rotated and flipped, we will take $G = D_6$ and X to be the set of possible configurations. The number of configurations $\lvert X \rvert = t^6$. Let us now consider the number of fixed points for each element of D_6. We label the vertices $x_1, \ldots, x_6$, where each x_i can be one of t colours. Being a fixed point of each rotational/ reflectional symmetry places certain conditions on the choice of x_i.

$$\begin{array}{cccc} & \circ x_3 & \circ x_2 & \\ \circ x_4 & & & \circ x_1 \\ & \circ x_5 & \circ x_6 & \end{array}$$

Then we get the following number of fixed points for each element of D_6. Each letter A, B, C, D denotes a distinct colour.

Element	$\lvert X^g \rvert$	Pattern
r_0	t^6	All patterns
r_1	t	$AAAAAA$
r_2	t^2	$ABABAB$
r_3	t^3	$ABCABC$
r_4	t^2	$ABABAB$
r_5	t	$AAAAAA$
s_0	t^4	$ABCDCB$
s_1	t^3	$AABCCB$
s_2	t^4	$ABACDC$
s_3	t^3	$ABBACC$
s_4	t^4	$ABCBAD$
s_5	t^3	$ABCCBA$

Hence we get

$$\text{Number of orbits} = \frac{1}{12}(t^6 + 3t^4 + 4t^3 + 2t^2 + 2t).$$

If $t = 1$, we get $\frac{1}{12}(1 + 3 + 4 + 2 + 2) = 1$ as expected, as there is only one necklace that can be made with six beads of the same colour.

References

[1] Chapter 29. Gallian, Joseph. Contemporary abstract algebra. Nelson Education, 2012.